Technical Paper:

Parameter Estimates

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1 Introduction

For the pricing of quantos and convertible bonds under two-factor models, certain parameters have to be estimated. In principle, this could be done by using parameters implied form market prices of liquid instruments. However, if one wants to price a, say, vanilla quanto and has to use market prices of quantos for doing so, this is not the appropriate approach. Hence, we suggest to combine implied parameters and historical estimates, as demonstrated in the following section.

2 Cross Currency Interest Rate Instruments

In the UnRisk PRICING ENGINE cross currency interest rate instruments can be valuated under the assumption of two factor dependency. The factors r_1 and r_2^* describe the interest rates in the two different considered currencies. The type of the used stochastic interest rate models is of General Hull & White, such that the stochastic differential equations read as:

$$dr_1 = (\eta_1(t) - \gamma_1(t) r_1) dt + \sigma_1(t) dW_1 \qquad (\gamma_1(t) > 0)$$

$$dr_2^* = (\eta_2(t) - \gamma_2(t) r_2^*) dt + \sigma_2(t) dW_2 \qquad (\gamma_2(t) > 0)$$

The instantaneous coefficient of correlation between the interest rate r_1 and r_2^* is denoted by ρ.
These equations describe, e.g., the evolution of the USD short rate r_1 and of the EUR short rate r_2^*. If we now assume that cash flows are realized in USD, the equation for the EUR short rate r_2^* must be adjusted so that it reflects the evolution from the point of view of a risk-neutral U.S. investor rather than a risk-neutral European investor.
According to Hull & White (1994) this is realized by using equation

$$dr_2 = (\eta_2(t) - \rho_X \sigma_2(t) \sigma_X - \gamma_2(t) r_2) dt + \sigma_2(t) dW_2 \qquad (\gamma_2(t) > 0)$$

r_2 now describes the EUR short rate from the point of view of a risk-neutral U.S. investor. The effect of moving from a EUR risk-neutral world to a USD risk-neutral world is to reduce the drift of r_2^* by $\rho_X \sigma_2(t) \sigma_X$. X is defined as the exchange rate (number of USD per EUR). σ_X denotes the volatility of the exchange rate and ρ_X the instantaneous coefficient of correlation between the exchange rate X and the EUR interest rate r_2^*.
In the valuation of cross currency instruments in the UnRisk PRICING ENGINE it is assumed, that the first interest rate model in the function call corresponds to the domestic currency, in which cash flows are realized!
The underlying partial differential equation to price instruments which depend on these two factors is written as:

$$\begin{aligned}
&\frac{\partial V}{\partial t}(r_1, r_2, t) + \frac{1}{2}\sigma_1^2(t)\frac{\partial^2 V}{\partial r_1^2}(r_1, r_2, t) + \frac{1}{2}\sigma_2^2(t)\frac{\partial^2 V}{\partial r_2^2}(r_1, r_2, t) \\
&+ \rho(t)\sigma_1(t)\sigma_2(t)\frac{\partial^2 V}{\partial r_1 \partial r_2}(r_1, r_2, t) + (\eta_1(t) - \gamma_1(t) r_1)\frac{\partial V}{\partial r_1}(r_1, r_2, t) \\
&+ (\eta_2(t) - \rho_X \sigma_2(t)\sigma_X - \gamma_2(t) r_2)\frac{\partial V}{\partial r_2}(r_1, r_2, t) - r_1 V(r_1, r_2, t) = 0.
\end{aligned}$$

which fits, of course, in the general formulation of our two-factor partial differential equation.
All necessary parameters concerning the General Hull & White model for the interest rates r_1 and r_2^* can be determined as usual. For the pricing of cross currency instrument three more parameters are necessary: ρ, ρ_X and σ_X.

2.1 Parameter Estimates

Under the assumption of given data sets for the exchange rate, $\hat{X} = (Log[X_2] - Log[X_1], ..., Log[X_N] - Log[X_{N-1}]$, and for interest rates, $\widehat{r_1} = (r_{1_2} - r_{1_1}, ..., r_{1_N} - r_{1_{N-1}}), \widehat{r_2} = (r_{2_2} - r_{2_1}, ..., r_{2_N} - r_{2_{N-1}})$, ρ, ρ_X and σ_X can be determined as follows:

$$\rho = \frac{Cov(\widehat{r_1}, \widehat{r_2^*})}{\sqrt{Var(\widehat{r_1})Var(\widehat{r_2^*})}}$$

$$\rho_X = \frac{Cov(\widehat{X}, \widehat{r_2^*})}{\sqrt{Var(\widehat{X})Var(\widehat{r_2^*})}}$$

$$\sigma_X = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(X_i - \overline{X}_i)^2} * \sqrt{\text{Number of Observation Dates per Year}}$$

where the mean level $\overline{x}$ of a given data set $x = (x_1, ..., x_N)$ is defined by the arithmetic mean:

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

The empirical variance of this set is:

$$Var(x) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2$$

and the empirical covariance for two scalar data sets $x = (x_1, ..., x_N)$ and $y = (y_1, ..., y_N)$ is:

$$Cov(x, y) = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})(y_i - \overline{y})$$

3 One factor short rate + one factor equity, Example: Convertible Bond

If we assume that the mathematical model for the pricing of, e.g., a convertible bond depends on two factors, one factor short rate, r, and one factor equity, S, we end up again in a two-factor model. The underlying stochastic differential equations are:

$$dr = (\eta(t) - \gamma(t)r)dt + \sigma_1(t)dW_1 \qquad (\gamma(t) > 0)$$

for the interest rate r, and

$$dS = \mu S dt + \sigma_2(t)dW_2$$

for the equity S. dW_1 and dW_2 are increments of Wiener processes. The corresponding random variables may be correlated and we assume that

$$E[dW_1, dW_2] = \rho(t)dt, \qquad -1 <= \rho(t) <= 1.$$

Arbitrage arguments again lead us to a two-dimensional partial differential equation of the form:

$$\begin{aligned}
&\frac{\partial V}{\partial t}(r,S,t) + \frac{1}{2}\sigma_1^2(t)\frac{\partial^2 V}{\partial r^2}(r,S,t) + \frac{1}{2}\sigma_2^2(t)S^2\frac{\partial^2 V}{\partial S^2}(r,S,t) \\
&+ \rho(t)\sigma_1(t)\sigma_2(t)S\frac{\partial^2 V}{\partial r \partial S}(r,S,t) + (\eta(t) - \gamma(t)r)\frac{\partial V}{\partial r}(r,S,t) \\
&+ rS\frac{\partial V}{\partial S}(r_1,S,t) - rV(r,S,t) = 0.
\end{aligned}$$

V = V(r,S,t) denotes the value of the convertible bond, which depends on r, S, and t.

3.1 Parameter estimates

All parameters concerning the one factor General Hull & White model for the interest rate can be determined as usual as well as the volatility σ_2 of the equity.
The correlation ρ of these two factors can be estimated again by the use of historical data. Under the assumption of a given data set for the equity price, $\hat{S} = (\text{Log}[S_2] - \text{Log}[S_1], ..., \text{Log}[S_N] - \text{Log}[S_{N-1}])$ and for the interest rates, $\hat{r} = (r_2 - r_1, ..., r_N - r_{N-1})$ the parameter ρ can be determined by:

$$\rho = \frac{Cov(\widehat{S}, \widehat{r})}{\sqrt{Var(\widehat{S})Var(\widehat{r})}}$$

The covariance and the variance are defined as in the previous section.

References

Hull, J. & White, A. (1994), 'Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models', *Journal of Derivatives* **2**(2), 37–48.