Andreas Binder

# How to kill your trees properly

Binomial trees might be great in the classroom, but in the real world they're so much firewood writes **Andreas Binder** of MathConsult

I don't like trees. At least I don't like them when used for the numerical solution of partial differential equations. From the theoretical point of view, binomial trees are quite appealing in teaching the concept of no-arbitrage. Nevertheless, from the numerical point of view, there are major drawbacks: You typically need a huge number of time steps to obtain a reasonable accuracy by binomial trees. This could be improved by trinomial trees, but the problem of instability remains.

To be more specific: Trinomial trees are explicit numerical schemes for typically parabolic differential equations, which may lead to severe stability problems. In the case of mean-reverting models, this is well known. Fiddling around with the branching of the tree makes the method stable again, but changes the domain of the partial differential equation and therefore its solution.

This article should give an overview how we at MathConsult work on taking the tree risk out of computational finance.

Linz is the industrial center of Austria and also one of the largest centers for industrial and applied mathematics on the worldwide scale.

## Ironmaking in blast furnaces

Iron is produced from iron ore, typically in blast furnaces with a typical furnace producing 2 or 3 million metric tons of metallic iron per year (and obviously also some industrial amount of carbon dioxide). Although iron has been smelted for at least 2500 years, little is known about what happens in the interior of a furnace and where it happens.

Modeling a blast furnace leads to systems of dozens of nonlinear transient partial differential equations, covering the flows of materials (iron and coke layers, reduction gas, additives controlling the basicity and viscosity of the slag), phase transitions, energy consumption and energy transport, chemical reactions and several more. A complete system of modeling equations may lead to a discretized system of several million spatial unknowns acting on different time scales.

At MathConsult, we have been working on mathematical modeling and numerical simulation of blast furnaces in several research projects.

Figure 1 shows the reduction degree (of iron ore) in a blast furnace. The blue stripes are the coke layers, where no reduction takes place. Red means metallic iron.

Would you like to rely on tree methods for simulating a chemical reactor?

## Reaction – convection – diffusion

When pricing structured instruments in quantitative finance under the assumption of mean-reverting models, you quite frequently obtain partial differential equations of reaction-convection-diffusion type. For example, this is the case for Hull-White, Black-Karasinski or certain types of electricity pricing models.

In Binder & Schatz (2004), we presented finite-elements and streamline diffusion as an advanced technique for solving reaction-convection-diffusion equations.

For example, let us start with a two-factor Hull-White interest rate model (see Hull & White, 1994)

$$\begin{aligned} dr &= [\theta(t) + u(t) - a(t)r(t)]dt + \sigma_1(t)dX_1 \\ du &= -b(t)u(t)dt + \sigma_2(t)dX_2 \end{aligned}$$

The first factor $r$ denotes the spot rate, the second factor $u$ some kind of long-term development of the interest rates. $a$ is the mean reversion speed of the spot rate $r$, $(\theta + u)/a$ its reversion level. The stochastic variable $u$ itself reverts to a level of zero at rate $b$. $dX_1$ and $dX_2$ are increments of Wiener processes with instantaneous correlation $\rho(t) \cdot \sigma_1$ and $\sigma_2$ are the volatilities.

No-arbitrage-arguments then lead to the Hull-White equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma_1(t)^2\frac{\partial^2 V}{\partial r^2} + \rho(t)\sigma_1(t)\sigma_2(t)\frac{\partial^2 V}{\partial r \partial u} + \frac{1}{2}\sigma_2(t)^2\frac{\partial^2 V}{\partial u^2} + (\theta(t) + u - a(t)\,r)\frac{\partial V}{\partial r} - b(t)u\frac{\partial V}{\partial u} - rV = 0,$$

which needs additional end and transition conditions. We will discuss the problem of boundary conditions, when restricting ourselves to a bounded calculation domain, below.

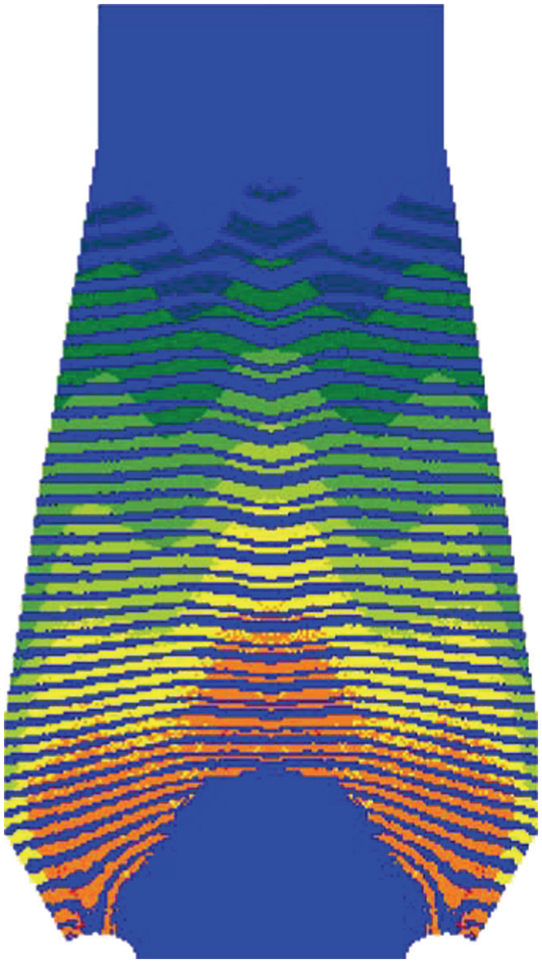

**Figure 1: Reduction degree in a blast furnace.**

The end and transition conditions describe the special shape of a financial contract, like coupons, callabilities and so on.

If we discretize time (for the ease of readability, I write down the fully implicit scheme) and multiply this equation by a test function $w$ living in a proper function space, we obtain after integration

Find $V^{n+1} \in U$ such that, for all $w \in U$,

$$
\begin{aligned}
&\int_\Omega \frac{V^{n+1} - V^n}{\Delta t} wd(r,u) \\
&- \int_\Omega \left( \frac{1}{2}\sigma_1^2 \frac{\partial V^{n+1}}{\partial r} + \frac{1}{2}\rho\sigma_1\sigma_2 \frac{\partial V^{n+1}}{\partial u} \right) \frac{\partial w}{\partial r} \\
&+ \left( \frac{1}{2}\rho\sigma_1\sigma_2 \frac{\partial V^{n+1}}{\partial r} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 V^{n+1}}{\partial u^2} \right) \frac{\partial w}{\partial u} d(r,u) \\
&+ \int_\Gamma \left( \frac{1}{2}\sigma_1^2 \frac{\partial V^{n+1}}{\partial r} + \frac{1}{2}\rho\sigma_1\sigma_2 \frac{\partial V^{n+1}}{\partial u} \right) w\, n_r \\
&+ \left( \frac{1}{2}\rho\sigma_1\sigma_2 \frac{\partial V^{n+1}}{\partial r} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 V^{n+1}}{\partial u^2} \right) w\, n_u ds \\
&+ \int_\Omega \left( (\theta + u - a\, r) \frac{\partial V^{n+1}}{\partial r} - bu \frac{\partial V^{n+1}}{\partial u} \right) wd(r,u) \\
&- \int_\Omega \left( r\, V^{n+1} \right) wd(r,u) = 0
\end{aligned}
$$

To obtain a finite-dimensional version, space discretization is necessary, for example by restricting the ansatz functions and the test functions to piecewise linear functions.

## The test bond and the model

We consider a 30 year coupon-bearing bond paying a fixed coupon of 5% per year. No credit risk as considered here. Under the swap curve of March 15, 2005, this bond has a fair value of 1.1086 on its start date.

A two factor Hull-White (Hull & White, 1994) with $a = 1.2$, $b = 0.03$, $\sigma 1 = 0.015$, $\sigma 2 = 0.01$, $\rho = 0.5$ is fitted to the swap curve by a piecewise constant $\theta(t)$.

## Calculation domain and discretisation

The domain of the Hull-White differential equation is – in principle – unbounded. For the numerical calculation, we restrict ourselves to a domain where influences from outside the domain should not play a role. We use a regular grid of 30 × 30 points, a time step of 0.05 years and Crank-

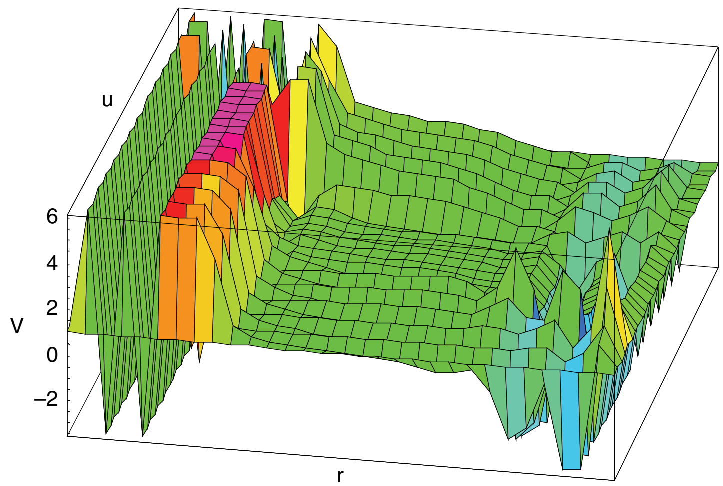


**Figure 2: Dirichlet boundary conditions, no streamline diffusion, shows severe oscillations.**

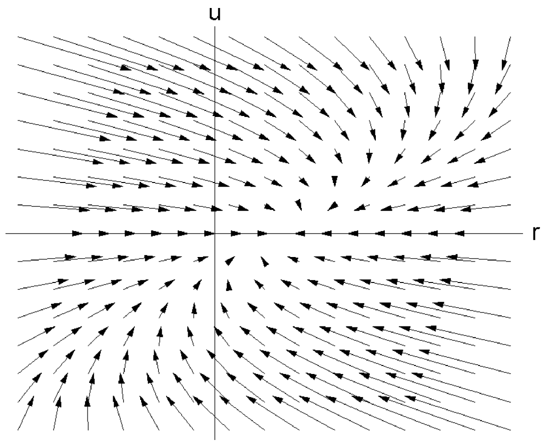


**Figure 3: Velocity field of the convective part.**

Nicolson time discretization. For Dirichlet boundary conditions $V = 1$ at the boundary, we obtain for different values of (r, u) at the start date of the bond:

The numerical value for our example bond is 1.1179, meaning an error of 93 basis points.

The heavy oscillations near the boundaries are not really a good argument in favor of finite elements. A slightly more careful investigation of the results shows that these oscillations arise in regions where convection plays an important role.

Streamline diffusion (as proposed in Binder & Schatz (2004) is equivalent to adding artificial diffusion along the streamlines of the flow. The choice of

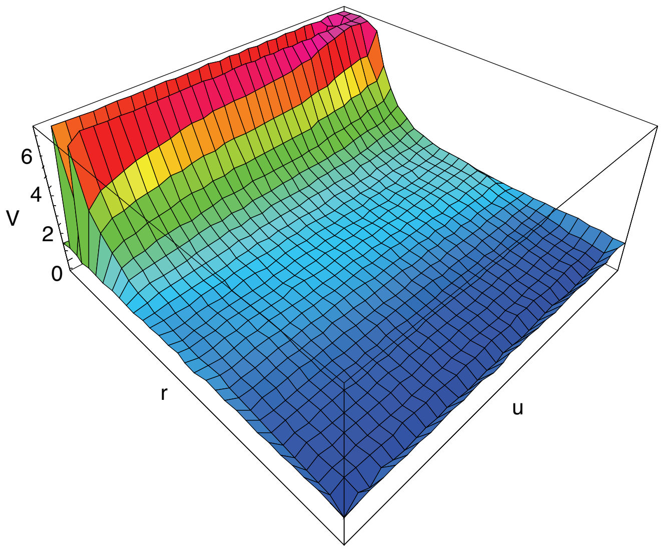


**Figure 4: Dirichlet boundary conditions, finite elements with streamline diffusion, value of the example bond: 1.1085.**

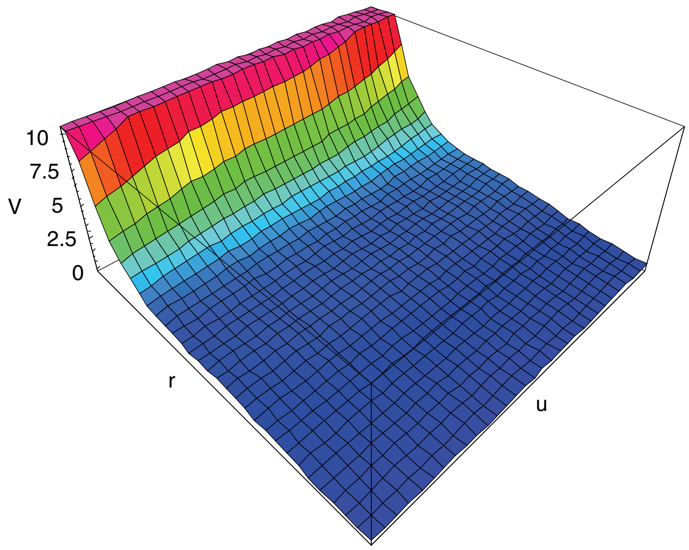


**Figure 5: Neumann boundary conditions, finite elements without streamline diffusion, value of the example bond: 1.1072.**

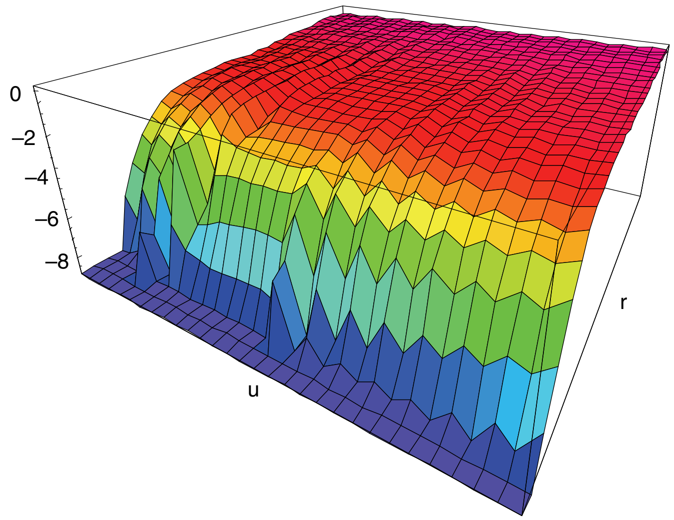


**Figure 6: Plot of first derivative with respect to r. Neumann conditions, no streamline diffusion.**

the smoothing parameter depends on the element size and on the convection /diffusion ratio. The advantage of this approach is that artificial diffusion is added only if and where necessary. If we again use Dirichlet conditions but now with the streamline diffusion added, we obtain

This result is already a very good one with an error of 1bp. Nevertheless, there are (due to the Dirichlet conditions) boundary layers which we would like to get rid of.

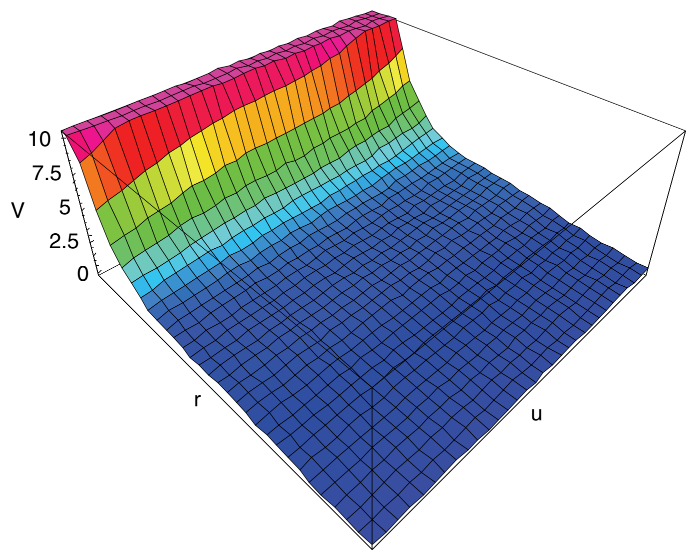


**Figure 7: Neumann boundary conditions with streamline diffusion. Value of the example bond: 1.1085. Error = 1bp.**

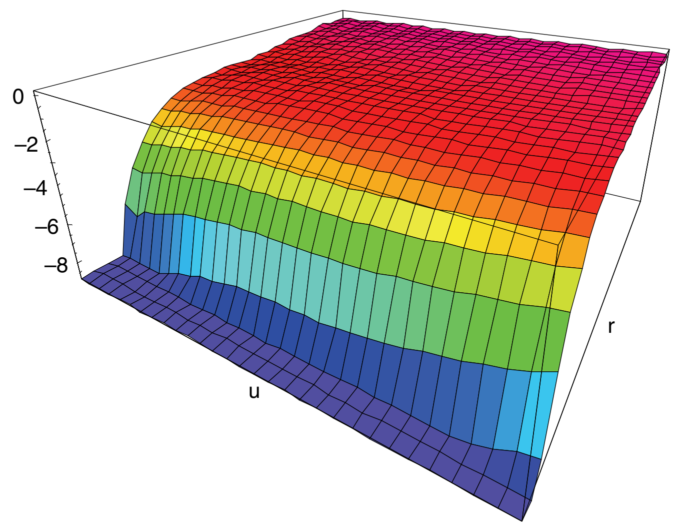


**Figure 8: Neumann boundary conditions with streamline diffusion. Smooth interest rate delta.**

With homogeneous Neumann boundary conditions, the boundary layers disappear:

For the results in Figure 5, we switched off the streamline diffusion again. The solution looks quite smooth, but the numerical error of 14 basis points might be improved. Using Neumann conditions removed the oscillations in the value, but they are still present in the first derivative with respect to the short rate.

The final two plots in Figures 7 and 8 are then the results for the value and the short rate delta when we apply Neumann conditions and streamline diffusion.

## Conclusion

With proper application of finite elements and streamline diffusion techniques, you have got the proper numerical techniques proven in industrial processes.

Trees are bad. Let's start the chain-saw!

### Links

If you are interested in more industrial mathematics at its best, visit http://www.ricam.oeaw.ac.at/media

### Author

*Andreas Binder received his PhD in Applied Mathematics (University of Linz) in 1991 for the numerical treatment of some problems in continuous casting of steel. He is CEO of MathConsult and CEO of the Industrial Mathematics Competence Center (IMCC) in Linz.*

*MathConsult has been developing advanced numerical methods for quantitative finance since 1997. The UnRisk PRICING ENGINE (current version is 2.5, the 12th release) came to the market in 2001 and serves customers in more than 20 countries.*

### FORTHCOMING

The next letter from Steel Town will be written by Heinz Engl. He is director of the Radon Institute for Compuatational and Applied Mathematics (RICAM). At the ICIAM (Zurich, July 2007) he will receive the ICIAM Pioneer Prize 2007, established for pioneering work introducing applied mathematical methods and scientific computing techniques to an industrial problem area or a new scientific field of applications.

## REFERENCES

Binder, A. and Schatz, A. (2004). Finite elements and streamline diffusion for the pricing of structured financial instruments, *Wilmott magazine*, November 2004, 97–103.

Duffy, D. (2004). A critique of the Crank Nicolson scheme strengths and weaknesses for financial instrument pricing, *Wilmott magazine*, July 2004, 68–76.

Engl, H.W., Gökler, G., Schatz, A., and Zeisel, H. (submitted). Modelling and numerics for the transient simulation of the blast furnace process.

Hull, J. and White, A. (1994). Numerical procedures for implementing term structure models II: two-factor models. *Journal of Derivatives*, 37–48.

Morton, K. W. (1996). Numerical Solution of Convection-Diffusion Problems. Chapman & Hall: London.

Stroustrup, B. (2000). *The C++ Programming Language*. Addison-Wesley: Reading, MA, USA.

Roos, H.-G., Stynes, M., and Tobiska, L. (1996). *Numerical Methods for Singularly Perturbed Differential Equations - Convection-Diffusion and Flow Problems*. Springer: Berlin.

UnRisk Manual: Available from www.unriskderivatives.com