Technical Paper:

Calibration of Equity Models

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1 Local Volatility Surface

1.1 Introduction

Consider put or call options on a given underlying with, for the time being, the same expiry date but with different strikes. In the classical Black-Scholes framework, you obtain the fair prices of these options by putting the same volatility into the Black-Scholes formulae for all these options. However, in reality it is observed that different strike prices imply different volatilities. If you plot these implied volatilities against the strike prices, the shape of the curve may be like a smile, which leads to the name "volatility smile". In equity markets, the implied volatilities often increase with decreasing strike prices. This may be explained by market participants' fear of market crashes which makes put options with low strikes more valuable. To recover the shape of the implied volatility, one possibility is that in the random walk for the underlying, the volatility does not only depend on t, but also on S.

$$dS(t) = \mu S(t)dt + \sigma(S,t)SdW$$

This is the concept of local volatility.
Identifying the local volatility surface from market prices of vanilla options is an ill-posed problem which needs to be treated with care in order to avoid numerical instabilities. Dupire (1994) showed that the identification of local volatility is as ill-posed as two times differentiation. For regions deep in the money or deep out of the money, it is even worse. Within UnRisk, the following sections describe how to get smooth surfaces for local volatility.

1.2 The mathematics behind the calibration

As a model for the evolution of a financial asset S_t, we consider a generalized Black-Scholes model proposed in Derman & Kani (1994), i.e.,

$$dS(t) = (r(t) - d(t))S(t)dt + \sigma(S_t,t)SdW_t \tag{1}$$

where r(t) denotes the interest rate, d(t) is the dividend yield, and σ(S,t) is the local volatility function. The stochastic process (1) is driven by a Brownian motion with increments dW_t. Using no-arbitrage arguments, it can be shown (cf. e.g., Wilmott (1998)) that the fair price C(K,T)of a European Call option with strike K and maturity T as a function of time t and value S of the underlying solves the Black-Scholes equation

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + (r-d)C_S - rC = 0 \tag{2}$$

$$C(T,S) = max(S-K,0) \tag{3}$$

Appropriate boundary conditions have to be added to (2), (3), in order to make the problem uniquely solvable.

1.3 The calibration problem

While the dividend yield d(t) and the interest rate r(t) can be assumed to be given by the market, or to be observable from other financial instruments, the local volatility function σ(S,t) has to be chosen in such a way that quoted market prices $C_*^{K,T}$ are matched, i.e.,

$$C_*^{K_i,T_j} = C^{K_i,T_j}(t = 0, S_0) \tag{4}$$

or all strikes K_i and maturities T_j.
Finding the volatility surface $\sigma(S,t)$ such that quoted market prices $C_*^{K_i,T_j}$ are matched by the solution to the Black-Scholes equation (2) corresponds to an inverse coefficient problem for a parabolic equation. Similar problems appear in many parameter identification problems in mathematical physics, cf., e.g., Binder et al. (1990), Burger et al. (1999), Engl & Kügler (2002) for some examples. Usually,parameter identification problems governed by partial differential equations are ill-posed in the following sense:

- a solution may not exist
- the solution (if it exists) may not be unique
- the solution does not depend on the data in a stable way (here option prices).

The last property implies that in principle arbitrarily large errors in the reconstructed volatility may arise from arbitrarily small perturbations in the data.
In order to solve the inverse problem (4) in a reasonable way, so-called regularization methods have to be used, cf. Engl et al. (1996, 2nd edition 2000) and B.Kaltenbacher et al. (2008) for an overview over regularization of inverse problems. A classical approach for solving nonlinear inverse problems is Tikhonov regularization: Denoting by F the operator which maps a volatility surface σ to the corresponding option values (via solution of (2),(3) for different strikes and maturities), a regularized solution (volatility surface) can be defined by

$$||F(\sigma) - C_*||^2 + \alpha||\sigma^2 - \sigma_*^2|| \to min \tag{5}$$

see Egger & Engl (2005). Here, $C_* = \{C_*^{K_i,T_j}\}$ denotes the collection of data (quoted prices),σ_* is an appropriate a-priori guess for a solution (e.g., the Black76 vola surface), and the regularization parameter α balances between fit to the data and stability. An important step towards a fast solution of the optimization problem (5) is that the option price as function of strike and maturity also satisfies the so-called Dupire equation, cf. Dupire (1994), namely

$$-C_T + \frac{\sigma^2(K,T)}{2} K^2 C_{KK} - (r - d)C_K - dC = 0 \tag{6}$$

Hence F(σ) can be evaluated by solving a single partial differential equation.
For the solution of the minimization problem (5), any reasonable descent algorithm can be used. Gradients, i.e., $F' * (C_* - F(\sigma))$, can be evaluated efficiently via adjoint methods, and hence the calculation of a gradient can be performed

by another solution of a partial differential equation. For a faster minimization of (5), Newton type methods are widely used. The regularized, linearized equation, which has to be solved in each step of a Newton type method reads

$$[F'(\sigma)^* F'(\sigma) + \alpha I]d\sigma = [F'(\sigma)]^*[C_* - F(\sigma)] - \alpha(\sigma^2 - \sigma_*^2) \tag{7}$$

An approximate solution of this equation can for instance be efficiently realized by some preconditioned conjugate gradient algorithm. For an iterative solution of (7) only application of $F'(\sigma)$ and $F'^*(\sigma)$ to certain elements of a vector space are required. This operations again can be realized by solving partial differential equations similar to (2) and (7).

1.4 Implementation

Within the UNRISK PRICING ENGINE, the following combination of methods is used to efficiently solve the calibration problem:

- The partial differential equations (2) respectively (7) are discretized by a finite difference method combined with a Cranck-Nicolson scheme for the time integration.
- In order to keep the dimension of the problem relatively small, the volatility surface is discretized on a rougher grid (taking into account the strikes and maturities of available option prices).
- Tikhonov regularization is used for regularization of the calibration problem. Additionally, a positivity constraint on the squared volatilities is imposed, i.e., $\sigma^2 \geq \sigma_{min} > 0$. The problem (time horizon, spot price of the underlying) is scaled to standard values. In this way, a reasonable choice of a regularization parameter can be made independent of the problem.
- For minimization of (5) a Newton-CG algorithm is used, i.e., the regularized Newton systems (7) are solved approximately by a conjugate gradient algorithm. The outer Newton iteration is stopped if
  - the discrepancy in observed and reconstructed option prices $D = max[(C_*^{K_i,T_j} - C^{K_i,T_j}(t = 0, S_0)]$ is sufficiently small,
  - a maximal number of iterations is reached,
  - the residual D is increased during the iteration.

1.5 Local FX Volatility Surface

The treatment of a local volatility surface for fx derivatives is the same as it is for equity derivatives. A local fx volatility surface is calibrated due to given implied volatilities of vanilla fx options with varying strike fx rates and different remaining lifetimes.

2 Heston Model

2.1 Introduction

The stock price process in the Heston Stochastic Volatility model (Heston (1993)) follows the Black Scholes SDE in which the volatility is behaving stochastically over time:

$$dS_t = (r_d(t) - r_f(t))S_t dt + \sqrt{v_t} S_t dW_t^1 \tag{8}$$

where r_d denotes the domestic yield curve, r_f denotes the foreign yield curve or the dividend yield curve and v denotes the stock price variance. The term $\sqrt{v_t}$ ensures that the volatility in the stock price process is non negative. In the Heston model the squared volatility is stochastic and follows the classical Cox-Ingersoll-Ross (CIR) process:

$$dv_t = \kappa(\theta - v_t^2)dt + \sigma\sqrt{v_t}dW_t^2 \qquad v_0 \geq 0 \tag{9}$$

dW_t^1 and dW_t^2 are two correlated standard Brownian motions such that

$$Cov(dW_t^1, dW_t^2) = \rho dt.$$

The variance process is always positive and cannot reach zero if the Feller condition holds, which is given by

$$2\kappa\theta > \sigma^2$$

$\theta > 0$ is the long term variance, of κ as the rate of mean reversion. With the constant volatility assumption in the Black Scholes model, one assumes that the underlying stock price process follows a lognormal stochastic process. The basic assumption in a stochastic volatility model is, that the volatility of the underlying stock is itself a random variable. There are two Brownian motions: one for the underlying stock and one for the volatility. The two processes of the Heston model are correlated, whereas the correlation parameter ρ describes the dependence of the two processes. In most cases the correlation parameter is negative, which means that increases/decreases in the stock price leads to decreases/increases in the variance process. Empirical studies have shown that an assets log-return distribution is non Gaussian. It is characterized by heavy tails and high peaks (leptokurtic). There is also empirical evidence that equity returns and implied volatility are correlated ("leverage effect"). In contrast, the Heston model can imply a number of different distributions, depending on the values of its parameters: ρ, which can be interpreted as the correlation between the log-returns and volatility of the asset, affects the heaviness of the tails and therefore the skewness of the distribution. σ, the volatility of volatility parameter is responsible for the kurtosis (peak) of the distribution. On the volatility surface sigma effects the intensity of the smile effect. κ, the mean reversion parameter represents the degree of volatility clustering, θ is the long term level of the variance process and v_0 is the initial variance of the underlying.

2.2 Heston model calibration

The Heston model has five independent parameters $(\kappa, \theta, \sigma, \rho, v_0)$, which have to be determined by calibrating the model to a market observed implied volatility surface for european options with different strikes and maturities. The volatilities are converted to a set of option prices $\{C_*(K_i,T_j)\}$ and the calibration process solves a least squares minimization problem and determines the model parameters which give the best fit to the given market data. Mathematically the squared differences between vanilla option market prices and that of the model are minimized over the parameter space, i.e.

$$\sum (C_*(K_i,T_j) - C_{Heston}(K_i,T_j))^2 \rightarrow min \tag{10}$$

with the Feller condition as nonlinear constraint

$$2\kappa\theta > \sigma^2$$

The resulting problem of parameter identification is ill-posed, so regularization techniques have to be used to obtain stable results.
Once a parameter set has been determined by the calibration routine, one can price other options (european vanilla options with different strikes or more exotic options like barriers).
The price to pay for more realistic models is the increased complexity of model calibration. The minimization problem is complicated to solve, because in general the the funcional to minimize is not convex, which poses some complications. The objective function does not have to have any special structure to guarantee that gradient-based methods lead to acceptable results.
Two groups of algorithms can be applied to solve these optimization problems. The first group are the locally convergent algorithms which will find a minimum but not necessarily the global one (e.g. Levenberg-Marquardt). The second group of algorithms are the globally convergent algorithms which should theoretically (run time going to infinity) be able to find the global minimum (Horst & Pardalos (1995)). The disadvantage of the second group is the enormous amount of computation time in comparison to the algorithms of the first group to obtain results. Once a risk neutral model is found, which reproduces the prices of liquid traded options quite well, this model is used to price exotic and illiquid options. The calibration function implemented in the UnRisk PRICING ENGINE searches in the five-dimensional parameter range for a good starting value for a gradient based optimization routine , and then solves the minimization problem using the Levenberg Marquadt algorithm. The algorithm determines the optimal direction moving downhill on the parameter manifold to the minimum of the objective function. The advantage of this method is that the calibration works reasonable fast, but one always has the risk to end up in a local minimum. As a consequence a good initial guess is crucial.

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