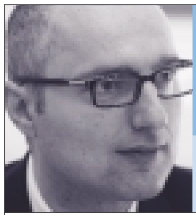


A clever handful is enough

Andreas Binder, binder@mathconsult.co.at

Introduction

It is common knowledge in risk management that movements of interest-rate curves can be mainly described by just a few factors (often named "shift," "twist," and "butterfly"). In this paper, we analyze if this knowledge is supported by evidence; we study what these factors look like and how many of them are needed to obtain a reasonable approximation. Finally, we discuss how these principal components could be applied for the fast calculation of key figures in quantitative risk management, especially for doing a historical value-at-risk (VaR) simulation.

Data Used

We start our analysis with daily EUR interest rate values (spot market, zero rates continuous compounding) between August 2000 and July 2007 (1,766 data sets) given for the curve points {overnight, 1 week, 3 months, 6m, 9m, 1 year, 2y, 3y, 4y, 5y, 7y, 10y, 15y, 20y, 25y, 30y, 50y}.

Figure 1 shows the shapes of EUR curves for the first business day of July for each of the years 2001 through 2007.

Interest-rate changes and their principal components

Hence, these interest-rate curves (or, as we do not prescribe a certain interpolation rule, to be more precise: interest-rate vectors) are points in a 17-dimensional space. As a change in the overnight rate has virtually no influence on the present value of future cash flows, we do not use it for our further calculations. Thus, we work with elements in a 16-dimensional space.

We calculated weekly changes of the given EUR curves and applied a very plain principal component analysis, meaning just the calculation of eigenvalues and eigenvectors of the Gramian matrix of the interest-rate increments. No fading memory effects were modeled or taken into account.

We did not care about different weights for different tenors of the interest rates. Therefore, in this first approach, the short end of the yield curve is more important, as the supporting points are more dense than at the far end.

The 16 principal components then had the following shapes:

For the calculation of these principal components of the increments, we used all data sets (between 2000 and 2007). The first three unit vectors exhibit the "shift, twist, butterly" behavior. Unit vector 1 explains 77 percent of interest-rate changes, 1 and 2 explain 92 percent , and 1, 2, and 3 explain 96.88 percent of the weekly interest-rate changes.

Figure 1. EUR curves for the first business day of July for each of the years 2001 through 2007.

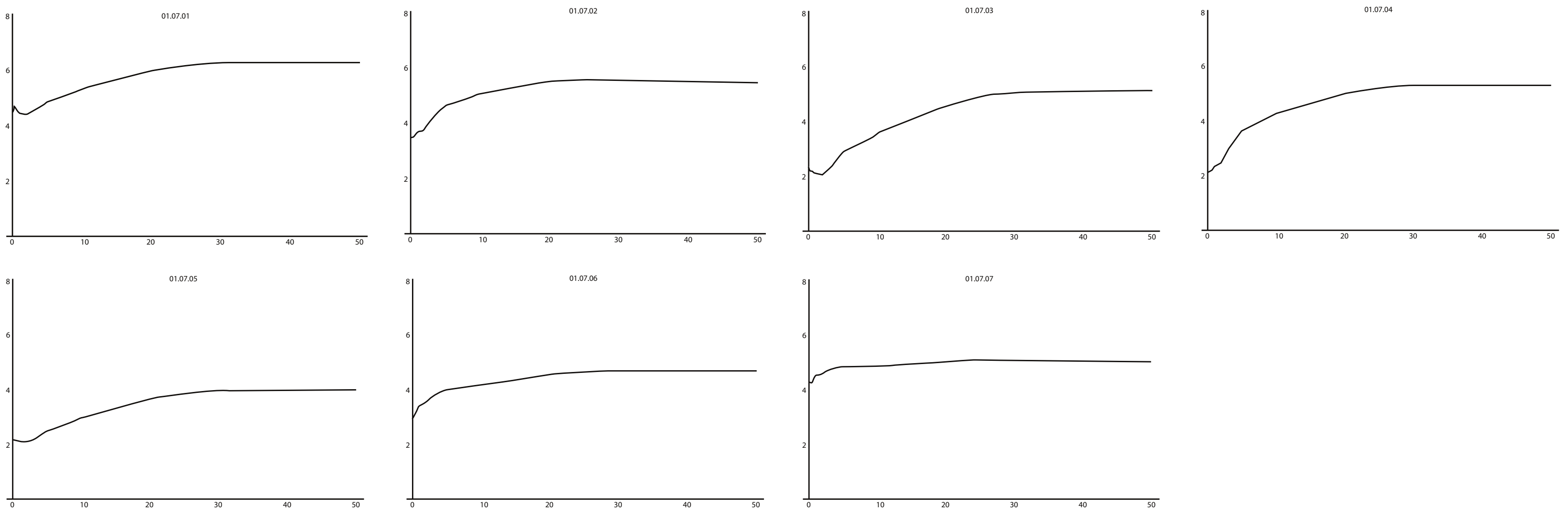
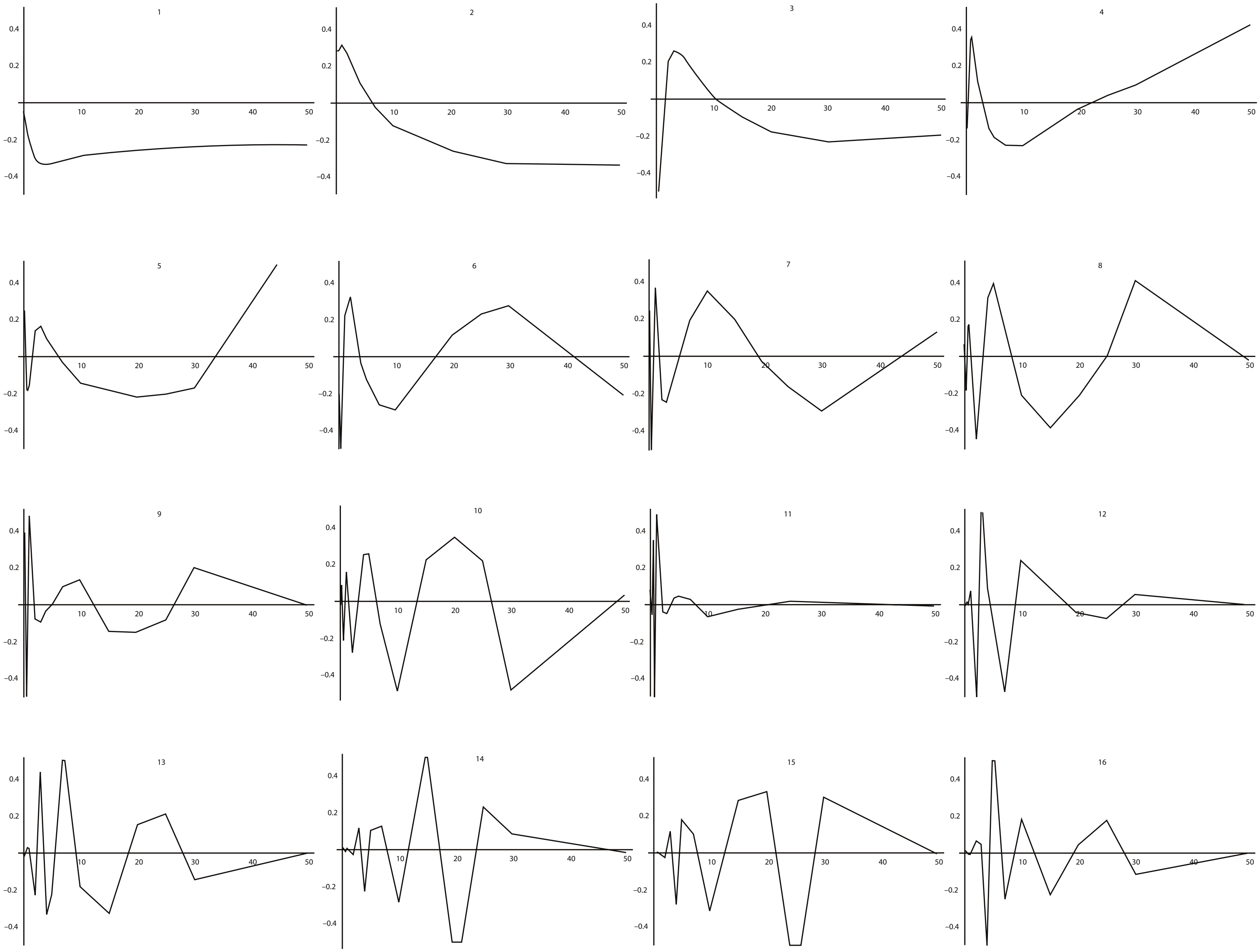


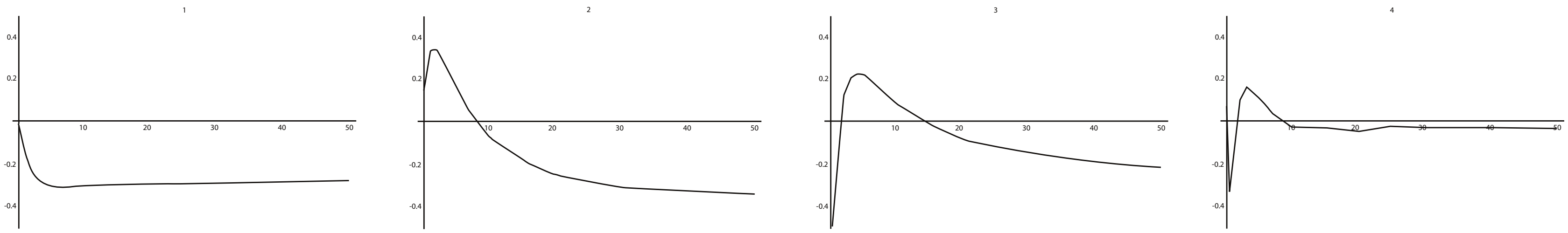
Figure 1. EUR curves for the first business day of July for each of the years 2001 through 2007 (continued).



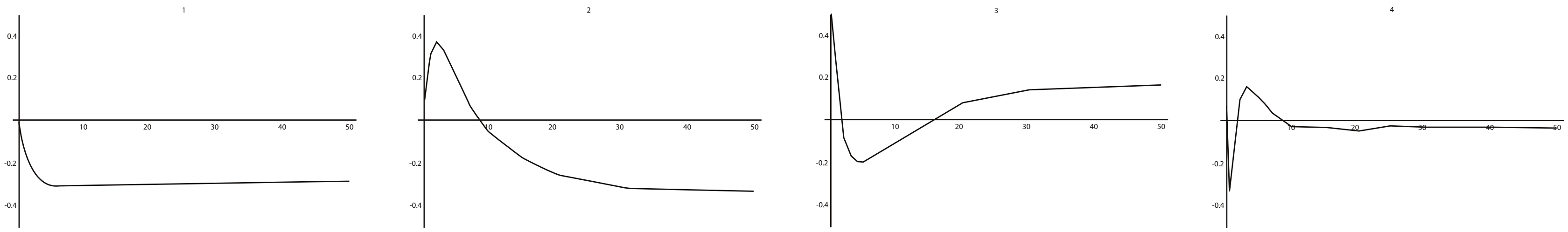
Robustness

For the above analysis, we used all available data sets. It turns out that if the observation window is reasonably long, the shapes of the main components more or less always look the same, as on the following page.

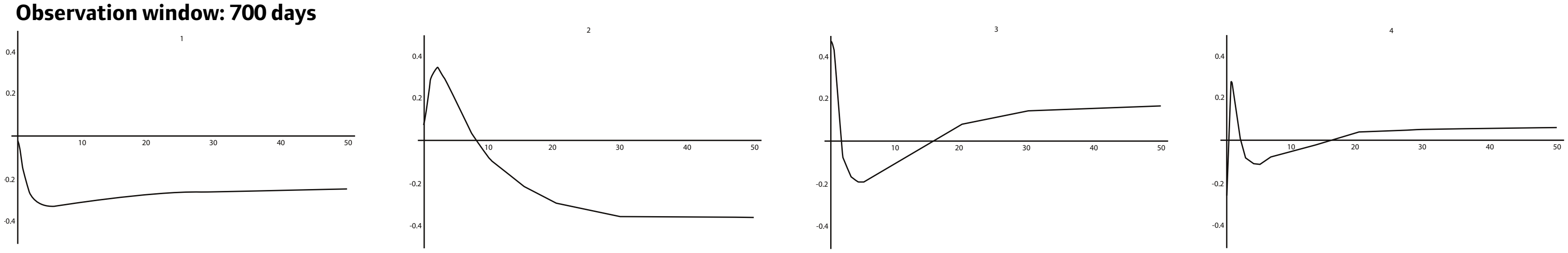
Observation window: 300 days



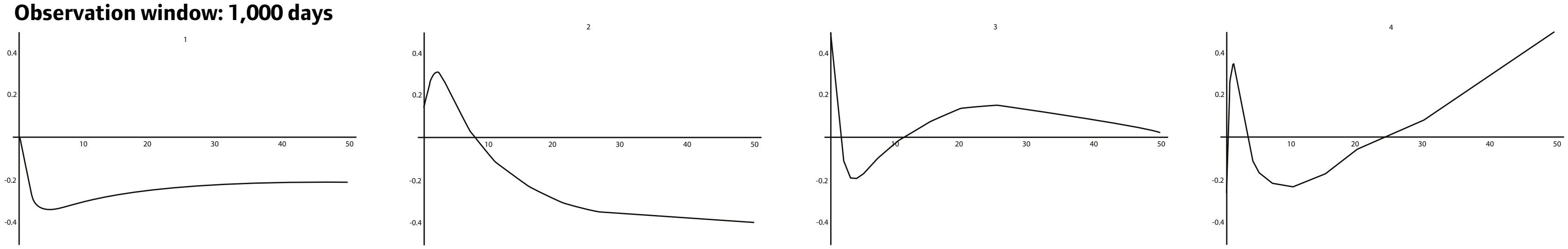
Observation window: 500 days



Observation window: 700 days

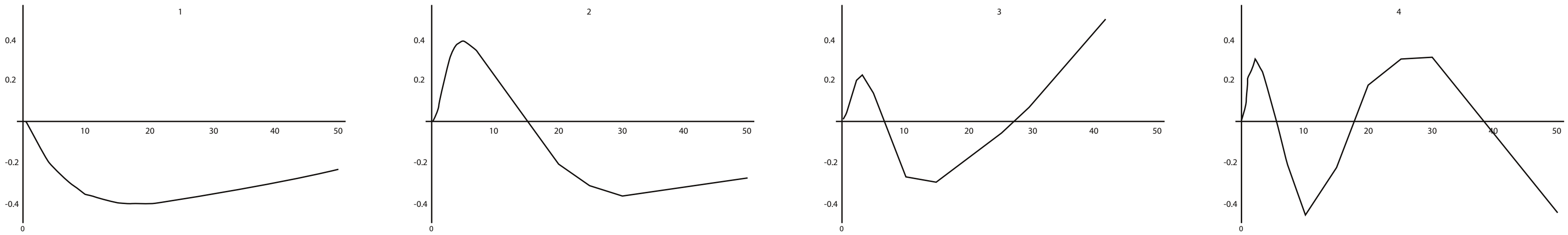


Observation window: 1,000 days



Discount factors or interest rates?

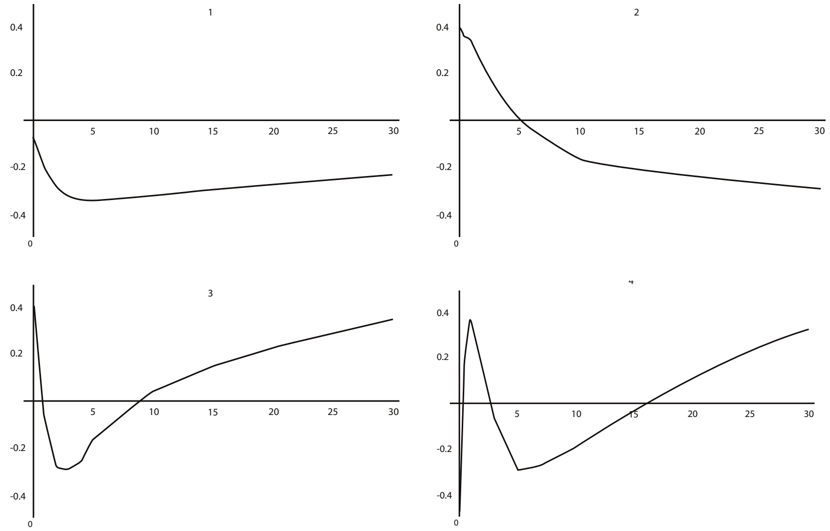
If we want to emphasize longer tenors of interest rates more than the shorter rates, we can use the vectors of change in the discount factors instead of the interest-rate changes:



As expected, the discount factors on the short end of the curve cannot change too much, and therefore the first principal components of the discount shifts start close to the origin.

Different currencies

We analyzed the interest-rate shifts for different currencies (USD, GBP, CHF, and JPY). It turns out that the principal components of the interest-rate changes exhibit the same qualitative behavior for all these currencies. As an example, we show the USD results.



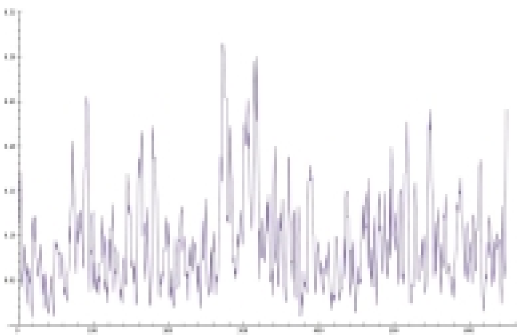
Quality of the projection to principal components

We applied the principal component analysis for the EUR yield curve increments to the first 1,000 data sets (2001–2004) and used the resulting principal components as a basis for the increments of the dates 2005 and later. We measured the norm of an increment by

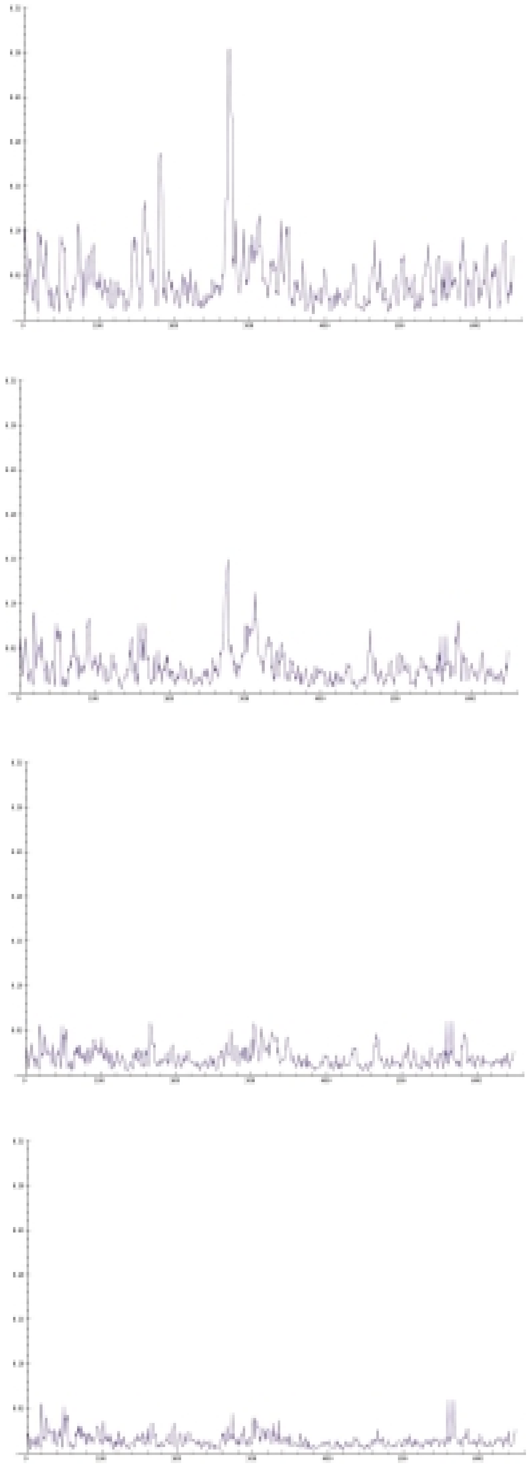
$$\|\Delta r\| = \sqrt{\frac{1}{16}\sum_{i=1}^{16}(\Delta r_i)^2}$$

and obtained the following:

Norm of weekly Increments for 650 business days. Scale is percent.



Norm of approximation error after filtering one, two, three, and four principal components. Scale is percent.



Application of these principal components to fast VaR calculation

Now assume that your risk manager wants you to carry out an historical VaR calculation for moderately complex financial instruments, which may be equipped with embedded and possibly multiple options like callabilities and which can therefore not be stripped into single cashflows. For the time being, assume that the value of such an instrument depends explicitly on interest-rate movements, but that volatility is assumed to be a parameter, which is not to be changed for the interest-rate VaR.

Then the straightforward way to a historical VaR consists of the following steps:

- Apply at least 250 historical changes of the interest-rate curve to today's yield curve.
- Calibrate the parameters of your preferred interest-rate model to the shifted yield curve data.
- Valuate all instruments in your pocket under these at least 250 scenarios.

Hence, if your portfolio consists of 1,000 instruments, this means that you have to carry out 250,000 valuations, which may take longer than a short coffee break.

However, as the valuation operators typically have smoothing properties, it makes sense to write — at least formally — for the value V of an instrument under the shifted curve $r + \mathrm{D}r$:

$$V(r + \mathrm{D}r) = V(r) + \mathrm{grad}\, V. \mathrm{D}r + \text{higher-order terms},$$

provided V is smooth.

Taylor expansion and VaR results

The valuation operator V does, for non-vanilla instruments, not depend directly on the yield curve but requires a typically ill-conditioned calibration process in a first step before the smoothing of the valuation is applied (Engl, 2007). Therefore, if we want to apply divided differences for the calculation of the gradient (and therefore can easily switch between different calibration and valuation routines), we should take into account that differences in the data with high frequencies will be amplified by the calibration process and may lead to oscillating results.

However, we have seen in the first section that the first principal components of interest-rate changes typically show low frequencies and therefore seem to be ideally suited as candidates for unit vectors in a transformed coordinate system.

For the ease of calculation, we calculated 95 percent and 99 percent historical VaRs (one-week horizon) by applying one-factor Hull white models to 1,000 weekly interest-rate shifts. We did this either by brute force (applying 1,000 curve-fitting and valuation routines) and by Taylor expansion for the first four, five, and six principal components, respectively. We valuated various callable interest-rate swaps (quarterly callable), reverse floating notes, CMS floating notes, and digital range accruals. The maturity of the instruments was up to 30 years, and the reverse floating notes were leveraged.

The results of the comparison were as follows:

- Typical errors between full historical 95 percent VaR and 95 percent VaR based on four principal components was between less than one basis point and up to ten basis points; for the 99 percent VaR was up to 30 basis points.
- The quality of the approximation for the digital range accrual VaR was lower due to the poorer quality of the Taylor approximation for the embedded digital options.
- There was no systematic increase in accuracy when applying five or six principal components instead of four.

Conclusions

Based on daily data sets between 2001 and 2007, we have analysed the main directions of interest-rate changes for the currencies EUR, USD, GBP, CHF, and JPY. The hypothesis that principal directions of interest-rate movements are shift, twist, and butterfly was confirmed. These principal components can and should be used as unit directions in models reduced in dimensionality. For the fast calculation of the historical VaR of moderately structured instruments, the approximation properties were extremely promising.

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I want to thank my colleague Johannes Fürst for providing the detailed examples.

Author

Andreas Binder received his PhD in applied mathematics (University of Linz) in 1991 for the numerical treatment of some problems in continuous casting of steel. He is CEO of MathConsult and CEO of the Industrial Mathematics Competence Center (IMCC) in Linz. MathConsult has been developing advanced numerical methods for quantitative finance since 1997. The UnRisk PRICING ENGINE (current version is 2.6, the 13th release) came to the market in 2001 and serves customers in more than 20 countries. The powerful grid-enabled UnRisk FACTORY will be released in spring 2008.

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