Technical Paper:

Valuation of Convertible Bonds

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1 Under a Black Scholes Model

The value of a callable / putable convertible bond is calculated by the use of Adaptive Integration. The valuation steps are:

1. build the equity price grid due to the given number of equity prices.

2. determine the life value (i.e. the value of the callable / putable convertible fixed rate bond if it has not been called, put or converted before) of the callable / putable convertible fixed rate bond at the maturity date T of the bond. It is assumed that the bond may neither be called nor put at the maturity date. If the bond is allowed to be converted into C equities the life value is given as

$$V[S[i],T] = \max(B, C * S[i])$$

   where B is the value of the fixed rate bond at the maturity date. If the bond pays a coupon $c[T]$, B is given as

$$B = \text{Redemption} + c[T]$$

   otherwise

$$B = \text{Redemption}$$

   If the fixed rate bond is not allowed to be converted at T, the life value is given by

$$V[S[i],T] = B$$

   The value of the convertible bond (CB) at time T is given as

$$CB[S[i],T] = V[S[i],T]$$

3. propagate back in time due to the given maximal length of a time step (i.e. building the grid in time direction) - we call it maxdt (the default value in the Mathematica Front End is 30 days). Let us assume we have got the values at date $t2$. The next date during the calculation is ($t2$ - maxdt) unless there is a key date between ($t2$ - maxdt) and $t2$. A key date is a dividend date, a Bermudan conversion date, a call date, a put date, a coupon date, the valuation day or one day after the valuation date. If one or more key dates are between ($t2$ - maxdt) and $t2$, the latest of these key dates is the next considered date. Let us call this date $t1$. The life value $V[S[i], t1]$ is obtained by

$$V[S[i],t1] = \exp(-r(t2-t1)) \int_{-\infty}^{+\infty} P[\xi] CB[S[i] + \xi + D[t2], t2] d\xi + c[t1]$$

   where $P[\xi]$ is the probability density (in the risk-free world) that $S[i]$ (given at $t1$) moves to $S[i] + \xi$ at $t2$ (see the section on Adaptive Integration in the paper "'Numberical Methods in UnRisk"'). Under the assumption of the geometric Brownian motion for the underlying equity price this density is the density of a lognormal distribution. r, y and σ are given by the forward values from $t1$ to $t2$ according to the corresponding yield and volatility curves. $c[t1]$ is the amount of the coupon payment if

$t1$ is a coupon date and 0 otherwise. $D[t2]$ is the amount of a discrete dividend payment if $t2$ is a dividend date and 0 otherwise. If the callable / putable convertible fixed rate bond is convertible at $t1$ the dirty value of the convertible bond is given as

$$CB[S[i], t1] = \max(C * S[i], \min(\text{Call}[t1], \max(\text{Put}[t1], V[S[i], t1])))$$

where Call[$t1$] is the considered call value if the bond is callable at $t1$ and Infinity otherwise, and Put[$t1$] is the considered put value if the bond is putable at $t1$ and 0 otherwise. This means that conversion overrules a call and a call overrules a put. If the bond is not convertible at $t1$ the value of the convertible bond is given by

$$CB[S[i], t1] = \min(Call[t1], \max(Put[t1], V[S[i], t1]))$$

4. the propagation backwards in time due to step c) is performed until the valuation date t is reached. The dirty convertible bond value for an equity price S is given as

$$CB[S, t] = \max(C * S, \min(\text{Call}[t], \max(\text{Put}[t], V[S, t])))$$

if the bond is convertible at the valuation date and

$$CB[S, t] = \min(\text{Call}[t], \max(\text{Put}[t], V[S, t]))$$

otherwise.

The values for delta, gamma and theta can easily be calculated during the same calculation procedure. For the calculation of vega, the volatility convexity and delta vega, the given volatility curve is shifted by $\pm 1\%$ (parallel shifts) and the value and the delta for these moved volatilities are calculated. If the vega is not needed it should not be calculated to save computation time. We also calculate the dirty and the clean value of the underlying fixed rate bond. The option value of the callable / putable convertible bond is then defined as

$$OV[S, t] = CB[S, t] - \text{Dirty Bond Value}[t]$$

and the clean convertible bond value is given by

$$\text{Clean } CB[S, t] = OV[S, t] + \text{Clean Bond Value}[t]$$

2 Under Black Scholes and General Hull & White Model

Under the assumption of two depending factors, one factor interest rate + one factor equity, the value of a callable / putable convertible fixed rate bond is calculated by the use of Finite Elements with Streamline Diffusion. The valuation algorithm is:

Step 1: Determination of the time discretization: Based on the maximal time step and on the given key dates the time discretization is determined in a way such that the actual time step does not exceed the maximal time step maxdt (the default value in the Mathematica Front End is 20 days) and all key dates are hit. Key dates are dividend dates, conversion dates, call dates, put dates, coupon dates, the valuation date, the settlement date and one day after the valuation date and after the settlement date. In additional dates at which one of the model parameter changes belong to the set of key dates.

Step 2: Determination of the space discretization: Depending on the volatilities of the two factors (the interest rate and the equity) and on the lifetime of the considered instrument the size of the discretization grid is determined as it is described in the paper "'Numerical Methods in UnRisk"' in chapter "'Streamline Diffusion"' (each direction represents one factor). The discretization grid itself consists of rectangles. The number of rectangles in each direction does not exceed the maximal number given by the function call option NumericalParameters2D, but it may be less if the required accuracy is obtained. The discretization grid is graded in a way such that it is finer near the center and coarser to the boundary.

Step 3: Determination of the life value on the grid: The life value is the value of the callable / putable convertible fixed rate bond if it has not been called, put or converted before. It is assumed that the bond may neither be called nor be put at the maturity date. If the bond is allowed to be converted into C equities the life value is given by:

$$V[r[i], S[i], T] = \max[B, C * S[i]]$$

where B is the value of the fixed rate bond at the maturity date T, which is

$$B = \text{Redemption} + c[T]$$

if the bond pays a coupon c[T] and

$$B = \text{Redemption}$$

otherwise. If the fixed rate bond is not allowed to be converted at T, the life value is given by

$$V[r[i], S[i], T] = B$$

If a discrete dividend D is paid at time t, across this dividend date the equity value drops by an amount of D but the bond value does not change. Mathematically we can write:

$$V[r[i], S[i], t^-] = V[r[i], S[i] - D, t^+]$$

So if a discrete dividend is paid at maturity the bond value at maturity is

$$CB[r[i], S[i], T] = V[r[i], S[i] - D, T]$$

and

$$CB[r[i], S[i], T] = V[r[i], S[i], T]$$

otherwise.

Step 4: Propagate back in time: Under the assumption that we know the value $CB^(j+1)$ at time t_{j+1} (starting with $t_{j+1} = T$, we want to determine the value at time t_j. The time step $\Delta t^j = t_{j+1} - t_j$ is given by the time discretization determined at the beginning of the algorithm. To calculate the dirty value at time t_j we have to solve the following partial differential equation for V^j numerically using the method of Finite Elements including Streamline Diffusion as upwind technique:

$$\begin{aligned}
&\frac{CB^{j+1} - V^j}{\Delta t^j} + \\
&\alpha\Bigg(\frac{1}{2} * \sigma_1^2[t_{j+1}]\frac{\partial^2 CB^{j+1}}{\partial r^2} + \rho[t_{j+1}]\sigma_1[t_{j+1}]\sigma_2[t_{j+1}]S\frac{\partial^2 CB^{j+1}}{\partial r \partial S} + \\
&\frac{1}{2} * \sigma_2^2[t_{j+1}]S^2\frac{\partial^2 CB^{j+1}}{\partial S^2} + (\nu[t_{j+1}] + \gamma[t_{j+1}]r)\frac{\partial CB^{j+1}}{\partial r} - \\
&+(r - d[t_{j+1}])S\frac{\partial CB^{j+1}}{\partial S} - rCB^{j+1}\Bigg) + \\
&(1-\alpha)\Bigg(\frac{1}{2} * \sigma_1^2[t_j]\frac{\partial^2 V^j}{\partial r^2} + \rho[t_j]\sigma_1[t_j]\sigma_2[t_j]S\frac{\partial^2 V^j}{\partial r \partial S} + \\
&\frac{1}{2} * \sigma_2^2[t_j]S^2\frac{\partial^2 V^j}{\partial S^2} + (\nu[t_j] - \gamma[t_j]r)\frac{\partial V^j}{\partial r} - \\
&(r - d[t_j])S\frac{\partial V^j}{\partial S} - rV^j\Bigg) = 0
\end{aligned}$$

d denotes the continuous equity yield. If the callable / putable convertible fixed rate bond is convertible at t_j the dirty value is given by

$$(CB^j)[r[i], S[i]] = \max[C{*}S[i], \min[\mathrm{Call}[t_j], max[\mathrm{Put}[t_j], (V^j)[r[i], S[i]] + C^j]]]$$

C^j is the amount of the coupon payment if t_j is a coupon date and 0 otherwise. Call$[t_j]$ is the considered call value if the bond is callable at t_j and infinity otherwise. Put[Subscript[t, j]] is the considered put value if the bond is putable at t_j and 0 otherwise. This means that conversion overrules a call and a call overrules a put. If the bond is not convertible at t_j the value of the convertible bond is given by:

$$(CB^j)[r[i], S[i]] = \min[\mathrm{Call}[t_j], \max[\mathrm{Put}[t_j], (V^j)[r[i], S[i]] + C^j]]$$

If a discrete dividend D^j is paid at t_j the bond value is

$$(CB^j)[r[i], S[i]] = (CB^j)[r[i], S[i] - D^j]$$

This jump condition requires interpolation of the calculated values at all grid points in the discretization shifted by $-D$. Step 4 is repeated ($t_{j+1} = t_j$) until the settlement date is reached!

Step 5: We know the spot rate r and the equity price S at the valuation date t, but we want to know the dirty value of the convertible bond at the settlement date. In order to obtain this value we have to solve the following problem between the settlement date and the valuation date:

$$
\begin{aligned}
&\frac{CB^{j+1} - V^j}{\Delta t^j} + \\
&\alpha\Bigg(\frac{1}{2} * \sigma_1^2[t_{j+1}]\frac{\partial^2 CB^{j+1}}{\partial r^2} + \rho[t_{j+1}]\sigma_1[t_{j+1}]\sigma_2[t_{j+1}]S\frac{\partial^2 CB^{j+1}}{\partial r \partial S} + \\
&\frac{1}{2} * \sigma_2^2[t_{j+1}]S^2\frac{\partial^2 CB^{j+1}}{\partial S^2} + (\nu[t_{j+1}] + \gamma[t_{j+1}]r)\frac{\partial CB^{j+1}}{\partial r} - \\
&+(r - d[t_{j+1}])S\frac{\partial CB^{j+1}}{\partial S} - rCB^{j+1}\Bigg) + \\
&(1-\alpha)\Bigg(\frac{1}{2} * \sigma_1^2[t_j]\frac{\partial^2 V^j}{\partial r^2} + \rho[t_j]\sigma_1[t_j]\sigma_2[t_j]S\frac{\partial^2 V^j}{\partial r \partial S} + \\
&\frac{1}{2} * \sigma_2^2[t_j]S^2\frac{\partial^2 V^j}{\partial S^2} + (\nu[t_j] - \gamma[t_j]r)\frac{\partial V^j}{\partial r} - \\
&(r - d[t_j])S\frac{\partial V^j}{\partial S}\Bigg) = 0
\end{aligned}
$$

This in principle is the same problem as in Step 4, but without discounting between the valuation date and the settlement date. The returned values of the pricing algorithm are: the dirty bond value, the clean bond value, the option value, the dirty convert value, the clean convert value, delta, gamma, option theta, bond carry, vega volatility, volatility convexity, and delta vega.

The dirty and the clean value of the underlying fixed rate bond are determined, of course, under the given General Hull & White interest rate model. The values for delta, gamma, theta, and the dirty bond value are calculated in the same procedure reusing the assembled system matrices as far as possible. In this way computation time can be saved. For the calculation of vega, of the volatility convexity, and of delta vega the dirty value and the delta are calculated under a shifted volatility curve by $\pm 1\%$. To save computation time the vega should only be calculated if it is really needed.