Technical Paper:

Calibration of Interest Rate Models

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1 General Hull & White Model

1.1 Introduction

The General Hull & White model is a one factor interest rate model of the form

$$dr = (\eta(t) - \gamma(t)r)dt + \sigma(t)dW \ (\gamma(t) > 0)$$

where $\eta(t)$ is the deterministic drift, $\gamma(t)$ is the reversion speed and $\sigma(t)$ is the Hull & White volatility. In the UnRisk PRICING ENGINE $\eta(t)$, $\gamma(t)$ and $\sigma(t)$ are assumed to be piecewise constant. Therefore the General Hull & White model can be considered as a piecewise Vasicek model.

1.2 Theoretical Background

We consider a generalized one-factor Hull & White model, where the short rate process is assumed to follow

$$dr = (\eta(t) - \gamma(t)r)dt + \sigma(t)dW \ (\gamma(t) > 0)$$

with dW being the increment of a Wiener process, $\sigma(t)$ being the volatility of the short rate process at time t, $\gamma(t)$ the mean reversion speed and $\eta(t)$ a function which allows the Hull & White model to fit the actual yield curve.

No-arbitrage arguments (cf., e.g. Wilmott (1998)) allow an equivalent formulation by means of a parabolic partial differential equation (backwards in time) for the value V of a bond or a derivative security

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial^2 t} + (\eta - \gamma r)\frac{\partial V}{\partial r} - rV = 0$$

Appropriate end conditions (payoff at maturity) and boundary conditions have to be formulated to make the pricing problem uniquely solvable. In order to be consistent with the market, one has to identify the Hull & White parameter functions (η, γ, σ) from market prices of liquid instruments. Here, we deal with the calibration of the one-factor Hull & White model, given the swap curve and matrices of Black 76 cap volatilities and / or Black 76 swaption volatilities.

1.3 Inverse and Ill-Posed Problems

The parameters of the Hull & White model, which are required as input for pricing purposes, cannot be observed directly because they refer to the future development of interest rates. Hence, one has to identify implied volatilities from market prices of liquid instruments. Mathematically speaking, this means the identification of the unknown parameters ($\eta(t)$, $\gamma(t)$, $\sigma(t)$) from market prices, and this means in particular the identification of diffusion coefficients in parabolic equations. This is an ill-posed problem which needs proper treatment. Examples of parameter identification problems in different contexts but with mathematical similarities can be found in Binder et al. (1990), Burger et al. (1999) and Engl & Kügler (2002).

A mathematical problem is said to be well-posed if

1. for all data, there exists a solution

2. the solution is unique, and

3. the solution depends continuously on the data.

In our case, condition 3) is violated, i.e., small perturbations in the (market) data may lead to arbitrarily large perturbations in the parameters of the interest rate model. For the stable treatment of inverse and ill-posed problems, so-called regularization techniques have to be applied. For an overview on these, see Engl et al. (1996, 2nd edition 2000). The classical approach to stabilize ill-posed problems is by Tikhonov regularization. Instead of solving F(x)=y, one solves the optimization problem

$$\min ||F(x) - y||^2 + \alpha||x - x^\star||^2$$

where $x^\star$ is an estimate for the solution x. In our case F would be the function, which maps the parameters of the interest rate model to the prices of caps and swaptions. The term $\alpha||x - x^\star||^2$ is a penalty term for the distance between x and $x^\star$. Other penalty terms are possible, for example terms which penalize the variation of the solution (bounded variation regularization, see Scherzer (2002)). A different approach are iterative regularization schemes like Landweber iteration (see Engl & Scherzer (2000)). In contrast to well-posed problems, it turns out that for ill-posed problems steepest-descent-like schemes under certain conditions converge faster than Newton-type schemes (Burger (2001)).

1.4 Identification of the Hull & White parameters by Regularization Techniques

We identify the parameter functions ($\eta(t)$, $\gamma(t)$, $\sigma(t)$) from an interest rate curve (money market and swap rates) and from matrices of cap and swaption prices for various strikes, expiries and maturities. We want to identify ($\eta(t)$, $\gamma(t)$, $\sigma(t)$) as piecewise constant functions with possible jumps of $\eta(t)$ at the terms of the interest rate curve and with possible jumps for $\gamma(t)$ and $\sigma(t)$ at the terms of the caps / swaptions. If $\eta(t)$, $\gamma(t)$, $\sigma(t)$ are known as piecewise constant functions, analytic solutions for zero bond prices and for caps, floors and swaptions are available and implemented in the UnRisk PRICING ENGINE. If $\gamma(t)$ and $\sigma(t)$ are given as piecewise constant functions, then $\eta(t)$ (as a piecewise constant function) can be identified from zero bond prices by solving a triangular linear system. Let $\eta(\gamma, \sigma)$ denote this drift function being consistent with the yield curve. The functions $\gamma(t)$ and $\sigma(t)$ to be identified should

1. lead to prices of caps / swaptions close to market prices, when applying an $(\eta(\gamma, \sigma), \gamma, \sigma)$ Hull & White model,

2. not show severe oscillatory behavior.

It turns out that a combination of Tikhonov regularization, bounded variation regularization and of a steepest descent iterative scheme leads to very good results. To be more specific: We minimize

$$\sum(\text{HullWhiteCapSwaptions}(\gamma, \sigma) - \text{MarketPrices})^2 + \alpha_1(\gamma - \gamma^\star) + \alpha_2(\sigma_{i+1} - \sigma_i)^2$$

and use a steepest descent algorithm with appropriate line search for minimization.

2 Hull & White 2 Factor Model

2.1 Introduction

In this section we consider an interest rate model, which is a generalization of the 2 factor model of Hull & White (see Hull & White (1994)). It incorporates a stochastic reversion level for the spot rate. The two factors are assumed to fulfill the following stochastic differential equations:

$$\begin{aligned} dr &= (\theta(t) + u - a(t)r)dt + \sigma_1(t)dW_1 \\ du &= -b(t)udt + \sigma_2(t)dW_2, \end{aligned}$$

a is the mean reversion speed of the spot rate r, $\theta + u$ its reversion level. The stochastic variable u itself reverts to a level of zero at rate b. dW_1 and dW_2 are increments of Wiener processes with instantaneous correlation $\rho(t)$, i.e.,

$$\mathbb{E}[dW_1 dW_2] = \rho(t)dt, \; -1 \leq \rho(t) \leq 1$$

σ_1 and σ_2 are the volatilities. All model parameters are assumed to be time dependent and piecewise constant.

This model provides a richer pattern of term structure movements and of volatility structures than the corresponding 1 factor models.

2.2 Theoretical Background

Starting from the above stochastic differential equations, no arbitrage arguments lead to an equivalent two-dimensional partial differential equation for the value $V(r, u, t)$ of a bond or a derivative security:

$$\begin{aligned} &\frac{\partial V}{\partial t}(r,u,t) + \\ &\frac{1}{2}\sigma_1^2(t)\frac{\partial^2 V}{\partial^2 r}(r,u,t) + \rho(t)\sigma_1(t)\sigma_2(t)\frac{\partial^2}{\partial r \partial u}(r,u,t) + \frac{1}{2}\sigma_2^2(t)\frac{\partial^2 V}{\partial u^2}(r,u,t) + \\ &(\theta(t) + u - a(t)r)\frac{\partial V}{\partial r}(r,u,t) - b(t)u\frac{\partial V}{\partial u}(r,u,t) - rV(r,u,t) = 0 \end{aligned}$$

Appropriate boundary and end conditions (payoff at maturity) have to be formulated to make the pricing problem uniquely solvable. The aim of the calibration is to identify the parameter functions (a, b, σ_1, σ_2, ρ, θ) from market prices of liquid instruments. The given partial differential equation let us conclude that identification of these parameters means to determine diffusion coefficients in parabolic equations. This is an ill-conditioned problem and requires sophisticated numerical methods, so-called regularization methods. Two popular classes of regularization methods are Tikhonov regularization and iterative techniques (see Engl et al. (1996, 2nd edition 2000)). In the two-factor case we use a combination of bounded variation regularization and regularization by iteration with "early stopping".

2.3 Identification of the Hull & White parameters by Regularization Techniques

We identify the parameter functions (a, b, σ_1, σ_2, ρ, θ) from an interest rate curve (money market and swap rates) and from matrices of cap and swaption prices for various strikes, expiries and maturities. The functions σ_1, σ_2, and θ are assumed to be piecewise constant with possible jumps of $\theta(t)$ at the terms of the interest rate curve and at the swaption expiries and with possible jumps of σ_1 and σ_2 at the swaption expiries. The parameters a, b and ρ are identified as constants. In the case that the parameter functions are given in this form we could derive analytic solutions for zero bond prices, for caps and floors, and for forward start swaptions, which are implemented in the UnRisk PRICING ENGINE. Knowing all parameters except the deterministic drift θ, θ can be determined by the use of zero bond prices. Let $\theta(a, b, \sigma_1, \sigma_2, \rho)$ denote this drift function being consistent with the yield curve. The functions $a(t)$, $b(t)$, $\sigma_1(t)$, $\sigma_2(t)$ and $\rho(t)$ should be determined such that they lead to prices of caps and swaptions close to market prices, when applying the corresponding Hull & White two-factor model and the functions are stable. So, we have to solve the following problem:

$$\sum(\text{HullWhiteCapSwaptions}[a, b, \sigma_1, \sigma_2, \rho] - \text{MarketPrices})^2 \rightarrow \min!$$

We solve this problem in two steps: In the first step we identify approximations for the reversion speed a and the correlation ρ by minimizing the error in the cap prices only. In the second step we minimize a combination of the error in the cap prices and the error in the swaption prices and identify the whole number of parameter functions, (a, b, σ_1, σ_2, ρ, θ) using the approximations of step one as starting values. The regularization is done by an iterative technique, a truncated Newton algorithm with the conjugate gradient method as inner iteration (see Hanke (1997)). In contrast to the calibration of the one-factor models the calibration of a Hull & White 2 factor model is only possible using both, cap and swaption data.

3 Black Karasinski Model

3.1 Introduction

The Black Karasinski model is a one factor interest rate model of the form

$$d \ln r = (\eta(t) - \gamma \ln r)dt + \sigma dW \qquad (\gamma > 0)$$

where $\eta(t), \gamma$ and σ are defined as deterministic drift, reversion speed and volatility of lognormal interest rates. In the UnRisk PRICING ENGINE $\eta(t)$ is assumed to be piecewise constant ,γ and σ are assumed to be positive constants.

3.2 Theoretical Background

We consider a one-factor Black Karasinski model, where the short rate process is assumed to follow

$$dlnr = (\eta(t) - \gamma lnr)dt + \sigma dW \qquad (\gamma > 0)$$

with dW being the increment of a Wiener process, σ being the volatility of the lognormal short rate process, γ the mean reversion speed and $\eta(t)$ a piecewise constant function which allows the Black Karasinski model to fit the actual yield curve.
No-arbitrage arguments (cf.,e.g. Wilmott (1998)) allow an equivalent formulation by means of a parabolic partial differential equation (backwards in time) for the value V of a bond or a derivative security

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 r^2 \frac{\partial^2 V}{\partial r^2} + (\eta(t) + \frac{1}{2}\sigma^2 - \gamma \ln r) r \frac{\partial V}{\partial r} - rV = 0$$

Appropriate end conditions (payoff at maturity) and boundary conditions have to be formulated to make the pricing problem uniquely solvable. The equivalent formulation of the partial differential equation in logarithmic variables (R=ln(r)) is

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 \frac{\partial^2 V}{\partial R^2} + (\eta(t) - \gamma R) \frac{\partial V}{\partial R} - e^R V = 0$$

In order to be consistent with the market, one has to identify the Black Karasinski parameter functions $\eta(t), \gamma$ and σ from market prices of liquid instruments. By letting η be the only time dependent function, we decide to exactly fit the current term structure of interest rates and to keep the other two parameters at our disposal for the calibration to cap and swaption data.

3.3 Inverse and Ill-Posed Problems

The parameters of the Black Karasinski model, which are required as input for pricing purposes, cannot be observed directly, because they refer to the future development of interest rates. Mathematically speaking, this means the identification of the unknown parameters $\eta(t), \gamma$ and σ from market prices, and this means in particular the identification of diffusion coefficients in parabolic equations. This is an inverse problem which is usually ill-posed and needs to be treated with sophisticated numerical methods, so called regularisation methods to get a stable solution of the problem. A mathematical problem is said to be well-posed if

- for all data, there exists a solution
- the solution is unique, and
- the solution depends continuously on the data.

In our case, the third condition is violated, i.e., small perturbations in the (market) data may lead to arbitrarily large perturbations in the parameters of the interest rate model. For the stable treatment of inverse and ill-posed problems, so-called regularization techniques have to be applied. For an overview on these, see Engl et al. (1996, 2nd edition 2000).
The classical approach to stabilize ill-posed problems is by Tikhonov regularization. Instead of solving F(x)=y, one solves the optimization problem

$$\text{minimize} ||F(x) - y||^2 + \alpha ||x - x^*||^2$$

where x^* is an estimate for the solution x.
In our case F would be the function, which maps the parameters of the interest rate model to the prices of zero coupon bonds, caps and swaptions. The term $\alpha||x - x^*||^2$ is a penalty term for the distance between x and x^*. Other penalty terms are possible, for example terms which penalize the variation of the solution (bounded variation regularization, see Scherzer (2002)).

3.4 Identification of the Black Karasinski parameters by Regularization Techniques

We identify the parameter functions $\eta(t), \gamma$ and σ from the maket prices of zero coupon bonds and form matrices of cap and swaption prices for various strikes, expiries and maturities. In a first step for given speed γ and volatility σ as constant functions of time, the piecewise constant drift $\eta(t)$ is determined to fit a given yield curve. The jumps of the piecewise constant function for the drift occur at the terms of the interest rate curve. This problem is called "curve fitting".

If $\eta(t), \gamma$ and σ are known, solutions for zero bond prices, caps and swaptions are available and implemented by the use of a finite difference method in the UnRisk PRICING ENGINE. Therefore we introduce a two dimensional grid for the parameter domain $[0, T]$x$[r_{min}, r_{max}]$, which spans from the valuation date to the maturity date in time direction and has the property, that all key dates (coupon dates, expiry date, ...) of the financial instrument to be valuated are hit exactly to guarantee accurate results. The time grid between these key dates is chosen to be equidistant. In direction of the underlying, the computational domain in our discrete implementation is chosen to be equidistant in the logarithm of r between a lower and an upper bound. Since the logarithm of zero is undefined, we use the lower bound r_{min} = 1e-5%. The upper bound r_{max} is chosen to be 100% because the sign of ln(r) does not change in this domain. For $r > 100\%$ the convection term is the partial differential equation would demand a different numerical treatment, For the most actively traded currencies, this limit is not really a restriction, because interest rates have been far away from 100% a long time. On this two-dimensional grid, the numerical solution of the Black Karasinski PDE for given parameter functions $(\eta(t), \gamma, \sigma)$ is calculated by the method of Crank-Nicolson. To guarantee stability in the solution and to avoid unrealistic oscillations, so called upwind techniques are used.

If $\gamma$and σ are given as constant functions, then $\eta(t)$ (as a piecewise constant function) can be identified from zero bond prices by solving the curve fitting problem. Therefore we define a nonlinear operator F which maps the drift function $\eta(t)$ to the corresponding prices of zero coupon bonds. Our aim is to solve the nonlinear equation $F(\eta(t)) = B^*$, where B^* is the vector of given zero coupon bond market data. This problem is equivalent to the minimization of the functional $||F(\eta(t)) - B^*||^2$. To obtain a stable solution of the inverse problem, we start with an initial guess x_0 of the solution and solve in every iteration step the linearized nonlinear equation $F'(x_k)(x - x_k) = -(F(x) - B^*)$ by using a truncated Tikhonov-CG algorithm. The solution of this equation is a good next approximation x_{k+1} to a solution. The iteration is stopped if the residual of the nonlinear equation is smaller than a given tolerance ϵ. Let

$(\eta(t), \gamma, \sigma)$ denote this drift function being consistent with the yield curve. The remaining constant parameters γ and σ to be identified should lead to prices of caps / swaptions close to market prices, when applying an $(\eta(\gamma, \sigma), \gamma, \sigma)$ Black Karasinski model. So we solve the following least squares problem:

$$\sum (\text{ModelPricesCapSwaptions}(\gamma, \sigma) - \text{MarketPricesCapSwaptions})^2 \rightarrow min!$$

4 LIBOR Market Model

4.1 Introduction

The LIBOR Market Model (LMM) is an n-dimensional (n factor) interest rate model in which the stochastic process for the forward interest rates $F_k(t)$ has the form (see Brigo & Mercurio 2006)

$$dF_k(t) = \sigma_k(t)F_k(t) \sum_{j=\beta(t)}^{k} \frac{\tau_j \rho_{j,k} \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t)F_k(t)dW_k(t) \qquad (1)$$

where $\sigma(t)$ and ρ are volatility and correlation of interest rates. In the UnRisk PRICING ENGINE $\sigma_k(t)$ follows the ansatz (e.g. Brigo & Mercurio 2006)

$$\sigma_k(t) = \psi_k((a(T_{k-1} - t) + b) \exp^{-c(T_{k-1}-t)} + d)$$

and ρ follows the ansatz (e.g. Rebonato 2006)

$$\rho_{i,j} = \exp^{-p_1(\exp^{-p_2 \min(i,j)})|i-j|}$$

whereas ψ_k, a, b, c, d and p_1, p_2 are constants with p_2 strictly positive. $dW_i(t)$ and $dW_j(t)$ are increments of Wiener processes with instantaneous correlation $\rho_{i,j}$, i.e.

$$dW(t)dW(t) = \rho dt$$

In the UnRisk PRICING ENGINE we are using the so called spot-measure dynamics representation (1) of the LMM. This representation has no known transition density, so that the equation from above needs to be discretised in order to perform simulations.

Applying Ito's Lemma, the equivalent log process for the interest rates is given by

$$dlnF_k(t) = \sigma_k(t) \sum_{j=\beta(t)}^{k} \frac{\tau_j \rho_{j,k} \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt - \frac{\sigma_k^2(t)}{2} dt + \sigma_k(t)dW_k(t) \qquad (2)$$

4.2 Theoretical Background

We consider the n factor LIBOR Market model, where the forward rate process is assumed to follow

$$dF_k(t) = \sigma_k(t)F_k(t) \sum_{j=\beta(t)}^{k} \frac{\tau_j \rho_{j,k} \sigma_j(t) F_j(t)}{1 + \tau_j F_j(t)} dt + \sigma_k(t)F_k(t)dW_k(t)$$

with σ being the volatility of the forward rate processes, τ the time difference, ρ the correlation matrix and dW the increment of an n-dimensional Wiener process with property

$$dW(t)dW(t) = \rho dt$$

The usual way to price bonds and other derivatives is done by Monte Carlo Simulation. For calibration a faster method is required. The quantity

$$\nu_{\alpha,\beta}^{LMM} = \frac{1}{T_\alpha} \sum_{i,j=\alpha+1}^{\beta} \frac{\omega_i(0)\omega_j(0)F_i(0)F_j(0)\rho_i, j}{S_{\alpha,\beta}^2(0)} \int_0^{T_\alpha} \sigma_i(t)\sigma_j(t)dt$$

is a good, fast and well tested approximation for Black76 swaption volatility (see Brigo & Mercurio 2006). If you put this quantity in Black's formula for swaptions, you can compute approximated swaption prices with the LMM. In order to be consistent with the market, one has to identify the LIBOR market model parameters (ψ, a, b, c, d and p_1, p_2 (see 2 and 3)) from market prices of liquid instruments. Due to the fact that the parameters cannot be observed in a direct way because they refer to the future development of interest rates, one has to use minimization methods to fit the unknown parameters to the given market data. In the UnRisk PRICING ENGINE we minimize the sum of the squares of the difference of approximated swaption volatilities (6) and given market swaption volatilities

$$\min \sum_{Swaptions} (\sigma_{\alpha,\beta}^{Mkt} - \nu_{\alpha,\beta}^{LMM})^2$$

by the use of a local minimization method (Levenberg-Marquardt) in connection with global optimization techniques to find acceptable starting points.

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